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Weakly Gibbsian Measures and Quasilocality: A Long-Range Pair-Interaction Counterexample

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We exhibit an example of a measure on a discrete and finite spin system whose conditional probabilities are given in terms of an almost everywhere absolutely summable potential but are discontinuous almost everywhere.

KEY WORDS: Gibbs measures; quasilocality; almost Gibbs; weakly Gibbs.

1. INTRODUCTION

This note is intended to provide a simple example of the non-equivalence of different notions introduced recently in order to generalize the standard Gibbs theory. We want to consider the difference between *weakly* Gibbsian measures and *almost* Gibbsian measures.

We shall work on a spin system whose configuration space is given by $\Omega = \{0, +1\}^{\mathbf{Z}^d}$. We denote by s an element of Ω and by s_A an element of $\Omega_A = \{0, +1\}^A$, for $A \subset \mathbf{Z}^d$. Consider potentials (interactions) $\Phi = (\Phi_X)$ which are families of functions

$$\Phi_X\colon \Omega_X \to \mathbf{R} \tag{1}$$

indexed by $X \in \mathscr{L}$, $|X| < \infty$. We say that a potential is $\bar{\Omega}$-pointwise absolutely summable, with $\bar{\Omega} \subset \Omega$, if

$$\sum_{X \ni x} |\Phi_X(s_X)| < \infty \qquad \forall x \in \mathbf{Z}^d, \quad \forall s \in \bar{\Omega} \tag{2}$$

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Then, if $\bar{\Omega}$ is in the tail-field, one may define a Hamiltonian in a finite volume V with boundary conditions $\bar{s} \in \bar{\Omega}$ by the usual formula:

$$H(s_V \mid \bar{s}_{V^c}) = \sum_{X \cap V \neq \varnothing} \Phi_X(s_{X \cap V} \vee \bar{s}_{X \cap V^c}) \tag{3}$$

where $s_V \in \Omega_V$, $\bar{s}_{V^c}$ is the restriction to V^c of $\bar{s} \in \bar{\Omega}$ and, for $X \cap Y = \varnothing$, $s_X \vee s_Y$ denotes the obvious configuration in $\Omega_{X \cup Y}$. We say that a measure on Ω is *weakly* Gibbsian if there exist a translation-invariant set $\bar{\Omega}$, and a $\bar{\Omega}$-pointwise absolutely summable interaction Φ such that $\mu(\bar{\Omega}) = 1$ and for μ there exists a version of the conditional probabilities that satisfy $\forall V \subset \mathbf{Z}^d$, $|V|$ finite, $\forall s_V \in \Omega_V$

$$\mu(s_V \mid \bar{s}_{V^c}) = \begin{cases} Z^{-1}(\bar{s}_{V^c}) \exp(-H(s_V \mid \bar{s}_{V^c})) & \text{for} \quad \bar{s} \in \bar{\Omega} \\ 0 & \text{for} \quad \bar{s} \notin \bar{\Omega} \end{cases} \tag{4}$$

Although the set $\bar{\Omega}$ is required to be translation invariant, this definition also include non-translation invariant measures.

A function f on Ω is said to be quasilocal at a certain s, if s is a point of continuity of f in the product topology on Ω, i.e.,

$$\forall \varepsilon > 0, \ \exists V(s) \text{ such that if } s'_{V(s)} = s_{V(s)}, \text{ we have, } |f(s) - f(s')| < \varepsilon \tag{5}$$

A function f is *essentially* discontinuous at s if

$$\exists \delta > 0, \ \forall V, \ \exists s', \text{ s.t. } s'_V = s_V \text{ and } \exists V'', \ V'' \supset V \text{ s.t. } \forall s''$$
$$\text{s.t. } s''_{V''} = s'_{V''}, \ |f(s'') - f(s)| > \delta \tag{6}$$

The difference between discontinuity and essential discontinuity is important in our context. Indeed, the points of essential discontinuity of a system of conditional probabilities can not be tranformed into points of continuity by modifying the conditional probabilities on a set of measure zero.

A measure μ on Ω is said to be *almost* Gibbsian[4] if there exists a version of its conditional probabilities that is quasilocal (as a function of the conditioning) μ-almost everywhere.

The quasilocality *everywhere* of the conditional probabilities is an important characterization of the measure due to a theorem by Kozlov and Sullivan.[3, 7] This theorem states that if a measure has a system of conditional probabilities that is quasilocal everywhere then, modulo a uniform positivity condition on the conditional probabilities, the measure is a Gibbs measure in the standard sense and the converse is also true.

The relationship between the two definitions introduced above (i.e., weakly and almost Gibbsian) is not clear a priori. A generalization of the result of Kozlov and Sullivan was obtained in ref. 4 for almost Gibbsian measures. Almost Gibbsian measures are also weakly Gibbsian. But, for example, the conditional probabilities of some measures obtained after a Renormalization Group transformation on the Ising model at low temperature have a non-empty set of points of essential discontinuity.[2] While the size of the measure of this set is unknown, it is shown, as in the Schonmann example,[6, 5] that those measures are weakly Gibbsian.[1]

We provide here an example of measure μ on Ω that is weakly Gibbsian but whose conditional probabilities are essentially discontinuous on a set of μ-measure 1. Our example is similar to the one given in ref. 4 but different in the sense that we are able to prove that the set of points of discontinuity is of measure 1 with respect to the measure under study, the form of our interaction being extremely simple (pair-interactions). Besides our model is defined in any dimension d (but is not translation invariant, as in ref. 4).

2. RESULTS

Consider the measure on Ω

$$\mu(ds) = \frac{\exp -H(s)}{Z}\, \mu_0(ds) \tag{7}$$

with μ_0 the product measure, $H(s)$ is formally given by

$$H(s) = \sum_{i,\, j \in \mathbf{Z}^d}^{\infty} 2^{|i| + |j|} s_i s_j \tag{8}$$

and obviously $Z = \int \exp -H(s)\, \mu_0(ds)$. Z is well-defined (finite although being defined in the infinite volume limit) because $\exp -H(s)$ is a limit of measurable functions and it is uniformly bounded on Ω, besides it is obviously non-zero.

Remark. Our argument in this note can be easily generalized to a Hamiltonian defined like in (8) but with $2^{|i| + |j|}$ replaced by ϕ_{ij} with ϕ_{ij} such that $\phi_{ij} \geqslant 0$ and $\phi_{ii} \to \infty$ as $|i| \to \infty$. The positivity of ϕ_{ij} is enough to guarantee the fact that $\exp -H(s)$ is uniformly bounded on Ω and thus that the partition function Z is well-defined.

It is easy to see that $H(s)$ is finite on the set

$$\Omega_g = \{ s \in \Omega \mid \exists \bar{V}(s) \text{ a cube}, \forall n \notin \bar{V}(s)\ s_n = 0 \} \tag{9}$$

Proposition 1.

$$\mu(\Omega_g) = 1 \tag{10}$$

Proof. Let us prove that $\mu(\Omega_g^c) = 0$. One has, $\Omega_g^c = \bigcap_{N=0}^{\infty} \Omega_N$ with,

$$\Omega_N = \{ s \in \Omega \mid \exists n \text{ s.t. } |n| \geqslant N \; s_n = 1 \} \tag{11}$$

and also,

$$\Omega_N \subset \bigcup_{n:\, |n| \geqslant N} \bar{\Omega}_n \tag{12}$$

with,

$$\bar{\Omega}_n = \{ s \in \Omega \mid s_n = 1 \} \tag{13}$$

Now, we write,

$$\mu(\bar{\Omega}_n) = \mu(s_n = 1) = \sum_{s:\, s_n = 1} \frac{\exp - H(s)}{Z} \tag{14}$$

and using the Hamiltonian (8), we see that this expression is bounded from above by

$$e^{-2^{2|n|}} \sum_{s:\, s_n = 0} \frac{\exp - H(s)}{Z} = e^{-2^{2|n|}} \mu(s_n = 0) \leqslant e^{-2^{2|n|}} \tag{15}$$

because we have $H(s) - H(\bar{s}) \geqslant 2^{2|n|}$ if s is such that $s_n = 1$ and $\bar{s}$ such that $\bar{s}_n = 0$, $\bar{s}_{\mathbf{Z}^d \setminus \{n\}} = s_{\mathbf{Z}^d \setminus \{n\}}$.

From the bound (15) it is easy to conclude that $\mu(\Omega_N) \leqslant c N^{d-1} e^{-2^{2N}}$ and thus that $\mu(\Omega_g^c) = 0$, which concludes the proof.

We can then easily compute the conditional distribution of the spin at the origin and get

$$\mu(s_0 \mid \bar{s}_{\{0\}^c}) = \frac{1}{1 + \exp(-(1 - 2s_0) \sum_{k \in \mathbf{Z}^d} 2^{|k|} \bar{s}_k)} \tag{16}$$

Obviously, this conditional probability is expressed in terms-of long-range two-body interactions and we may define the relative energy function of the spin at the origin.

$$h_0(s) = (1 - 2s_0) \sum_{k \in \mathbf{Z}^d} 2^{|k|} s_k \tag{17}$$

Obviously, just as $H(s)$, it is (absolutely) summable on Ω_g. To conclude, we need to show that this function is not only discontinuous but essentially discontinuous on Ω_g.

Proposition 2. h_0 is essentially discontinuous on Ω_g in the product topology on Ω.

Proof. From the definition of h_0 in (17) and Ω_g in (9), it is easy to see that if $s \in \Omega_g$ and $s' \in \Omega_g$ are such that $s_{\bar{V}(s)} = s'_{\bar{V}(s)}$ and $s_k \neq s'_k$, then $|h_0(s) - h_0(s')| \geqslant 2^{|k|}$ $(k \in \bar{V}^c(s))$. It is then easy to see that h_0 is essentially discontinuous on Ω_g; for any $s \in \Omega_g$, in (6) take $\delta = 1$ and $\forall V$, choose V'' and s' as follows, take V'' a finite (connected) set such that $V'' \supset (V \cup \bar{V}(s))$ and s' a configuration such that $s'_{V \cup \bar{V}(s)} = s_{V \cup \bar{V}(s)}$ and $s'_{V'' \setminus (V \cup \bar{V}(s))} \neq s_{V'' \setminus (V \cup \bar{V}(s))}$. Then for any configuration s'' such that $s''_{V''} = s'_{V''}$, it is clear that one has $|h_0(s'') - h_0(s)| > 1$.

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